

# ON THE DIAMETER OF THE COMMUTING GRAPH OF THE FULL MATRIX RING OVER THE REAL NUMBERS

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**ABSTRACT.** In a recent paper C. Miguel proved that the diameter of the commuting graph of the matrix ring  $M_n(\mathbb{R})$  is equal to 4 either if  $n = 3$  or  $n > 4$ . But the case  $n = 4$  remained open, since the diameter could be 4 or 5. In this work we close the problem showing that also in this case the diameter is equal to 4.

## 1. INTRODUCTION

For a ring  $R$ , the *commuting graph* of  $R$ , denoted by  $\Gamma(R)$ , is a simple undirected graph whose vertices are all non-central elements of  $R$ , and two distinct vertices  $a$  and  $b$  are adjacent if and only if  $ab = ba$ . The commuting graph was introduced in [1] and has been extensively studied in recent years by several authors [2, 3, 4, 5, 6, 7, 12, 13].

In a graph  $G$ , a path  $\mathcal{P}$  is a sequence of distinct vertices  $(v_1, \dots, v_k)$  such that every two consecutive vertices are adjacent. The number  $k - 1$  is called the length of  $\mathcal{P}$ . For two vertices  $u$  and  $v$  in a graph  $G$ , the distance between  $u$  and  $v$ , denoted by  $d(u, v)$ , is the length of the shortest path between  $u$  and  $v$ , if such a path exists. Otherwise, we define  $d(u, v) = \infty$ . The diameter of a graph  $G$  is defined

$$\text{diam}(G) = \sup\{d(u, v) : u \text{ and } v \text{ are distinct vertices of } G\}.$$

A graph  $G$  is called connected if there exists a path between every two distinct vertices of  $G$ .

Much research has been conducted regarding the diameter of commuting graphs of certain classes of rings [3, 7, 8, 10]. In the case of matrix rings over an algebraically closed field  $\mathbb{F}$ ,  $M_n(\mathbb{F})$ , it was proved [3] that the commuting graph with  $n > 2$  is connected and its diameter is always equal to four; while if  $n = 2$  the commuting graph is always disconnected [5]. On the other hand, if the field  $\mathbb{F}$  is not algebraically closed, the commuting graph  $\Gamma(M_n(\mathbb{F}))$  may be disconnected for an arbitrarily large integer  $n$  [4]. However, for any field  $\mathbb{F}$  and  $n \geq 3$ , if  $\Gamma(M_n(\mathbb{F}))$  is connected, then the diameter is between four and six [3]. Moreover, this diameter is conjectured to be at most 5 and if  $n = p$  is prime it is proved that the diameter is, in fact, 4. Quite recently, C. Miguel [11] has verified this conjecture in the case  $\mathbb{F} = \mathbb{R}$  proving the following result.

**Theorem 1.** *Let  $n \geq 3$  be any integer. Then,  $\text{diam}(\Gamma(M_n(\mathbb{R}))) = 4$  for  $n \neq 4$  and  $4 \leq \text{diam}(\Gamma(M_4(\mathbb{R}))) \leq 5$ .*

Unfortunately, this result left open the question whether  $\text{diam}(\Gamma(M_4(\mathbb{R})))$  is 4 or 5. In this paper we solve this open problem. Namely, we will prove the following result.

**Theorem 2.** *For every  $n \geq 3$ ,  $\text{diam}(\Gamma(M_n(\mathbb{R}))) = 4$ .*

## 2. ON THE DIAMETER OF $\Gamma(M_n(\mathbb{R}))$

Before we proceed, let us introduce some notation. If  $a, b \in \mathbb{R}$ , we define the matrix  $A_{a,b}$  as

$$A_{a,b} := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Now, given two matrices  $X, Y \in M_2(\mathbb{R})$ , we define

$$X \oplus Y := \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in M_4(\mathbb{R}).$$

Finally, two matrices  $A, B \in M_2(\mathbb{R})$  are similar ( $A \sim B$ ) if there exists a regular matrix  $P$  such that  $P^{-1}AP = B$ .

As we have pointed out in the introduction, in [11, Theorem 1.1.] it is proved that the diameter of  $\Gamma(M_n(\mathbb{R}))$  is equal to 4 if  $n \geq 3, n \neq 4$  and that  $4 \leq \text{diam}(\Gamma(M_4(\mathbb{R}))) \leq 5$ . The proof given in that paper relies on the possible forms of the Jordan canonical form of a real matrix. In particular, it is proved that the distance between two matrices  $A, B \in M_4(\mathbb{R})$  is at most 4 unless we are in the situation where  $A$  and  $B$  have no real eigenvalues and only one of them is diagonalizable over  $\mathbb{C}$ . In other words, the case when

$$A \sim \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix}, \quad B \sim \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix}.$$

The following result will provide us the main tool to prove that the distance between  $A$  and  $B$  is at most 4 also in the previous setting. It is true for any division ring  $D$ .

**Proposition 1.** *Let  $A, B \in M_n(D)$  matrices such that  $A^2 = A$  and  $B^2 = 0$ . Then, there exists a non-scalar matrix commuting with both  $A$  and  $B$ .*

*Proof.* Since  $A^2 = A$ ; i.e.,  $A(I - A) = (I - A)A = 0$ , then one of nullity  $A$  or nullity  $(I - A)$  is at least  $n/2$ . Since  $I - A$  is also idempotent and a matrix commutes with  $A$  if and only if it commutes with  $I - A$  we can assume that nullity  $A \geq n/2$ . On the other hand, since  $B^2 = 0$ , it follows that nullity  $B \geq n/2$ .

Now, if  $\text{Ker}L_A \cap \text{Ker}L_B \neq \{0\}$  and  $\text{Ker}R_A \cap \text{Ker}R_B \neq \{0\}$  we can apply [3, Lemma 4] and the result follows. Hence we assume that  $\text{Ker}L_A \cap \text{Ker}L_B = \{0\}$  (if it was  $\text{Ker}R_A \cap \text{Ker}R_B = \{0\}$  we could consider  $A^t$  and  $B^t$ ). Note that, in these conditions,  $n = 2r$  and nullity  $A$  and nullity  $B$  are equal to  $r$ .

Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be bases for  $\text{Ker}L_A$  and  $\text{Ker}L_B$ , respectively, and consider  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$  a basis for  $D^n$ . Since  $A$  is idempotent, it follows that  $D^n = \text{Ker}L_A \oplus \text{Im}L_A$ .

We want to construct the matrix of  $L_A$  in the basis  $\mathcal{B}$ . To do so, if  $v \in \mathcal{B}_2$ , we write  $v = a + a'$  with  $a \in \text{Ker}L_A$  and  $a' \in \text{Im}L_A$ . Hence,  $Av = Aa + Aa' = 0 + A(Aa'') = Aa'' = a' = -a + v$ . Since it is clear that  $Av = 0$  for every  $v \in \mathcal{B}_1$  we get that the matrix of  $L_A$  in the basis  $\mathcal{B}$  is of the form

$$\begin{pmatrix} 0 & A' \\ 0 & I_r \end{pmatrix},$$

with  $A' \in M_r(D)$ .

Now we want to construct the matrix of  $L_B$  in the basis  $\mathcal{B}$ . Clearly  $Bv = 0$  for every  $v \in \mathcal{B}_2$ . Now, let  $w \in \mathcal{B}_1$ . Then,  $Bw = w_1 + w_2$  with  $w_1 \in \text{Ker} L_A$  and  $w_2 \in \text{Ker} L_B$ . Hence,  $0 = B^2w = Bw_1$  and  $w_1 \in \text{Ker} L_A \cap \text{Ker} L_B = \{0\}$ . Thus, the matrix of  $L_B$  in the basis  $\mathcal{B}$  is of the form:

$$\begin{pmatrix} 0 & 0 \\ B' & 0 \end{pmatrix},$$

with  $A' \in M_r(D)$ .

As a consequence of the previous work we can find a regular matrix  $P$  such that:

$$PAP^{-1} = \begin{pmatrix} 0 & A' \\ 0 & I_r \end{pmatrix}, \quad PBP^{-1} = \begin{pmatrix} 0 & 0 \\ B' & 0 \end{pmatrix}.$$

Now, if  $A'B' \neq B'A'$  we can consider the matrix

$$P^{-1}(A'B' \oplus B'A')P = P^{-1} \begin{pmatrix} A'B' & 0 \\ 0 & B'A' \end{pmatrix} P,$$

which is clearly non-scalar and commutes with  $A$  and  $B$ . On the other hand, if  $A'$  and  $B'$  commute, we can find a non-scalar matrix  $S \in M_r(D)$  commuting with both  $A'$  and  $B'$ . Therefore  $P^{-1}(S \oplus S)P$  commutes with both  $A$  and  $B$  and the proof is complete.  $\square$

In addition to this result, we will also need the following technical lemmata.

**Lemma 1.** *If  $A \sim \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix}$ , then there exists an idempotent non-scalar matrix  $M$  such that  $AM = MA$ .*

*Proof.*  $A = P^{-1} \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix} P$  for some regular  $P \in M_4(\mathbb{R})$ . Hence, it is enough to consider  $M = P^{-1} \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix} P$ .  $\square$

**Lemma 2.** *If  $B \sim \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix}$ , then there exists a non-scalar matrix  $N$  such that  $N^2 = 0$  and  $BN = NB$ .*

*Proof.*  $B = P^{-1} \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix} P$  for some regular  $P \in M_4(\mathbb{R})$ . Hence, it is enough to consider  $N = P^{-1} \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix} P$ .  $\square$

We are now in the condition to prove the main result of the paper.

**Theorem 3.** *The diameter of  $\Gamma(M_4(\mathbb{R}))$  is four.*

*Proof.* In [11] it was proved that  $d(A, B) \leq 4$  for every  $A, B \in M_4(\mathbb{R})$  unless  $A \sim \begin{pmatrix} A_{a,b} & 0 \\ 0 & A_{c,d} \end{pmatrix}$  and  $B \sim \begin{pmatrix} A_{s,t} & I_2 \\ 0 & A_{s,t} \end{pmatrix}$ . Hence, we only focus on this case.

By Lemma 2 there exists an idempotent non-scalar matrix  $M$ , such that  $AM = MA$ . Also, by Lemma 2, there exists a non-scalar matrix  $N$  such that  $N^2 = 0$  and  $NB = BN$ . Finally, Proposition 1 implies that there exists a non-scalar matrix  $X$  that commutes both with  $M$  and  $N$ .

Thus, we have found a path  $(A, M, X, N, B)$  of length 4 connecting  $A$  and  $B$  and the result follows.  $\square$

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